New non-arithmetic lattices in $\mathrm{SU}(2,1)$

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- **No**, if $\mathbb{R}$-Rank$(G) \geqslant 2$ (Margulis).

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Group:	$\mathrm{SO}(n,1)$	$\mathrm{SU}(n,1)$	$\mathrm{Sp}(n,1)$	$F_4^{(-20)}$
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Minor open question: Do there exist infinitely many (non-commensurable) non-arithmetic lattices in $\mathrm{SU}(2,1)$?

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We constructed Dirichlet domains (numerically, with M. Deraux's program) for lots of these groups, which led us to:

Conjecture (DPP)

At least 11 of the sporadic groups are lattices, 4 C and 7 NC.

We also conjecture that almost all others are non-discrete (for 3 groups we don't conjecture anything). So far the conjecture has been established in most cases.

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The six groups $\Gamma(2\pi/p, \overline{\sigma_4})$ *(*$p = 3, 4, 5, 6, 8, 12$*) are lattices.*

Note that: one of these ($p = 3$) is the arithmetic one, the others are all non-arithmetic and new (3 C and 2 NC). The proof is by construction of a fundamental domain in $\mathrm{H}^2_{\mathbb{C}}$, so we also get presentations, volumes,...

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From this formula it is clear that $\mathrm{PU}(n, 1)$ acts by isometries on $\mathrm{H}^n_{\mathbb{C}}$ (where $\mathrm{U}(n, 1) < \mathrm{GL}(n + 1, \mathbb{C})$ is the subgroup preserving $\langle \cdot, \cdot \rangle$).

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In fact: $\mathrm{Isom}^+(\mathrm{H}^n_{\mathbb{C}}) = \mathrm{PU}(n, 1)$, and $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}}) = \mathrm{PU}(n, 1) \ltimes \mathbb{Z}/2$ (complex conjugation).

Totally geodesic subspaces: The only totally geodesic subspaces of $\mathrm{H}^n_{\mathbb{C}}$ are the projective images of complex linear subspaces (copies of $\mathrm{H}^k_{\mathbb{C}} \subset \mathrm{H}^n_{\mathbb{C}}$) and of totally real subspaces (copies of $\mathrm{H}^k_{\mathbb{R}} \subset \mathrm{H}^n_{\mathbb{C}}$).

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Important remark: Complex reflections may have arbitrary order (even infinite...)

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- Only finitely many of the $\Gamma(p,t)$ are discrete; the discrete ones are lattices.

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Theorem (Parker-P.)

Let R_1 be a complex reflection of order p and J a regular elliptic isometry of order 3 in $\mathrm{PU}(2,1)$. Suppose that $R_1 J$ and $R_1 R_2 = R_1 J R_1 J^{-1}$ are elliptic or parabolic. If the group $\Gamma = \langle R_1, J \rangle$ is discrete then one of the following is true:

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Mostow's lattices correspond to $\tau = e^{i\phi}$ for some angle ϕ; subgroups of Mostow's lattices to $\tau = e^{2i\phi} + e^{-i\phi}$ for some angle ϕ, and sporadic groups are those for which τ takes one of the 18 values $\{\sigma_1, \overline{\sigma_1}, ..., \sigma_9, \overline{\sigma_9}\}$ where the σ_i are given in the following list:

$$
\begin{array}{ll}
\sigma_1 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/4) & \sigma_2 := e^{i\pi/3} + e^{-i\pi/6}\, 2\cos(\pi/5) \\
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\sigma_5 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/5) & \sigma_6 := e^{2\pi i/9} + e^{-\pi i/9}\, 2\cos(4\pi/5) \\
\sigma_7 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(2\pi/7) & \sigma_8 := e^{2\pi i/9} + e^{-i\pi/9}\, 2\cos(4\pi/7) \\
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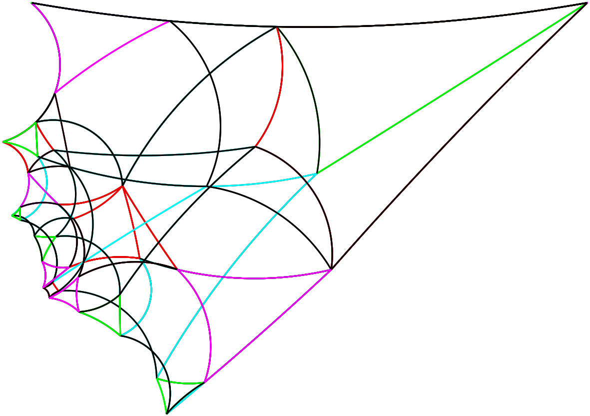
Therefore, for each value of $p \geqslant 3$, we have a finite number of groups to study, the $\Gamma(2\pi/p, \sigma_i)$ and $\Gamma(2\pi/p, \overline{\sigma_i})$ which are hyperbolic (i.e. preserve a form of signature (2,1)).

Figure : A view of the domain E for $p = 12$



Theorem (DPP)

Let $p \geqslant 3$*,* $R_1 \in \mathrm{SU}(2,1)$ *be a complex reflection through angle* $2\pi/p$ *and* $J \in \mathrm{SU}(2,1)$ *be a regular elliptic map of order* 3 *Suppose that* $\tau = \mathrm{Tr}(R_1 J) = \overline{\sigma_4} = -(1 + i\sqrt{7})/2$*.*

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The group $\langle R_1,\, J\rangle$ is a lattice whenever c and d are both integers, possibly infinity, that is when $p = 3,\, 4,\, 5,\, 6,\, 8,\, 12$.
Moreover, writing $R_2 = JR_1J^{-1}$ and $R_3 = JR_2J^{-1} = J^{-1}R_1J$, this group has presentation

$$\left\langle R_1,\, R_2,\, R_3,\, J \;\middle|\; \begin{array}{c} R_1^p = J^3 = (R_1J)^7 = id, \\ R_2 = JR_1J^{-1},\ R_3 = J^{-1}R_1J, \\ (R_1R_2)^2 = (R_2R_1)^2, \\ (R_1R_2)^{2c} = (R_1R_2R_3R_2^{-1})^{3d} = id \end{array} \right\rangle.$$

Bisectors:

Given 2 distinct points $p_1, p_2 \in \mathrm{H}^n_{\mathbb{C}}$, the *bisector* equidistant from p_1, p_2 is:

$$B(p_1, p_2) = \{p \in \mathrm{H}^n_{\mathbb{C}} | d(p, p_1) = d(p, p_2)\}.$$

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The intersection between a bisector B and a geodesic $g \not\subset B$ may contain 0, 1 or 2 points.

Description of the domains D and E:

We construct 2 related polyhedra in $\mathrm{H}^2_{\mathbb{C}}$. D will be a fundamental domain for the lattice Γ, and E will be a fundamental domain for the action of Γ modulo $\langle P \rangle$, where $P = R_1 J$ has order 7.

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E is constructed as follows: start with 4 bisectors $\mathcal{R}^{\pm}$ and $\mathcal{S}^{\pm}$, with $R_1(\mathcal{R}^+) = \mathcal{R}^-$ and $S_1(\mathcal{S}^+) = \mathcal{S}^-$.

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Proposition

E is cell-homeomorphic to a convex polytope in $\mathbb{R}^4$ (with some vertices removed when Γ is NC).

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If B_1 and B_2 are 2 coequidistant bisectors, then $B_1 \cap B_2$ is a (non-totally geodesic) smooth disk. Moreover, there exists a unique bisector $B_3 \neq B_1, B_2$ containing it.

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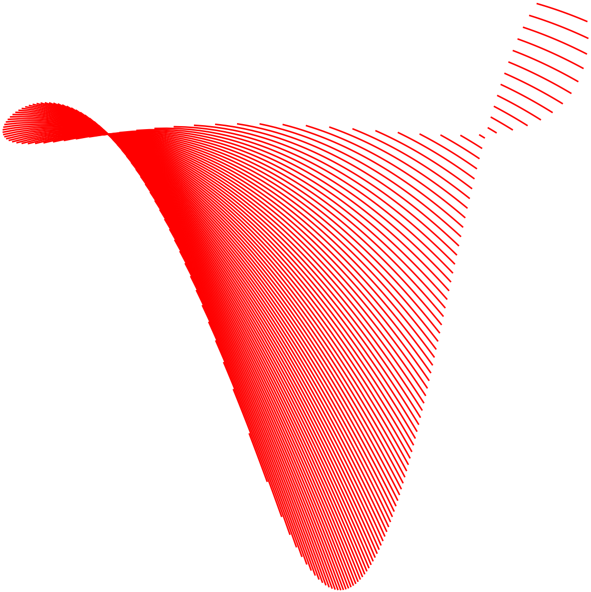
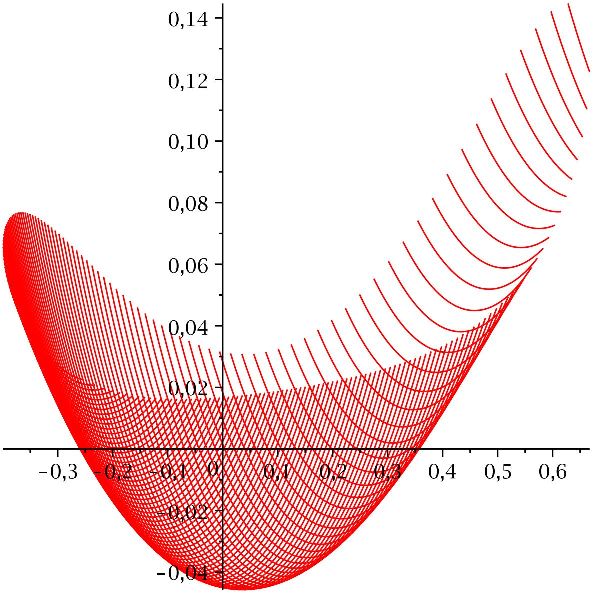
Proposition

All 2-faces of E are contained in Giraud disks or complex lines.

Bad projections of Giraud disks:




0,14
0,12
0,10
0,08
0,06
0,04
0,02
-0,3
-0,2
-0,1
0
0,1
0,2
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Corollary

(1) *The 6 groups* $\Gamma = \Gamma(2\pi/p, \bar{\sigma}_4)$ *with* $p = 3, 4, 5, 6, 8, 12$ *lie in different commensurability classes.*

(2) *The 6 groups* $\Gamma = \Gamma(2\pi/p, \bar{\sigma}_4)$ *with* $p = 3, 4, 5, 6, 8, 12$ *are not commensurable to any Deligne-Mostow lattice.*

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Under an additional assumption on the Galois conjugates ${}^{\varphi}H$ of the form (obtained by applying field automorphisms $\varphi \in \mathrm{Gal}(K)$ to the entries of the representative matrix of H), the group $SU(H, \mathcal{O}_K)$ is indeed discrete.

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2. *Suppose* $\Gamma \subset \mathrm{SU}(H; \mathcal{O}_E)$ *is a lattice. Then* Γ *is arithmetic if and only if for all* $\varphi \in \mathrm{Gal}(F)$ *not inducing the identity on* F*, the form* ${}^{\varphi}H$ *is definite.*

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Note that when the group Γ as in the Proposition is non-arithmetic, it necessarily has infinite index in $SU(H,\mathcal{O}_K)$ (which is non-discrete in $SU(H)$).

Definition: A Poincaré polyhedron is a smooth polyhedron D in X with codimension one faces T_i such that

1. The codimension one faces are paired by a set Δ of isometries of X which respect the cell structure (the side-pairing transformations). We assume that if $\gamma \in \Delta$ then $\gamma^{-1} \in \Delta$.
2. For every $\gamma_{ij} \in \Delta$ such that $T_i = \gamma_{ij} T_j$ then $\gamma_{ij} D \cap D = T_i$.

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Remark: If $T_i = T_j$, that is if a side-pairing maps one side to itself then we impose, moreover, that γ_{ij} be of order two and call it a reflection. We refer to the relation $\gamma_{ij}^2 = 1$ as a reflection relation.

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Cycles: Let T_1 be an $(n-1)$-face and F_1 be an $(n-2)$-face contained in T_1. Let T_1' be the other $(n-1)$-face containing F_1. Let T_2 be the $(n-1)$-face paired to T_1' by $g_1 \in \Delta$ and $F_2 = g_1(F_1)$. Again, there exists only one $(n-1)$-face containing F_2 which we call T_2'. We define recursively g_i and F_i, so that $g_{i-1} \circ \cdots \circ g_1(F_1) = F_i$.

Definition: *Cyclic* is the condition that for each pair (F_1, T_1)(an $(n-2)$-face contained in an $(n-1)$-face), there exists $r \geqslant 1$ such that, in the construction above, $g_r \circ \cdots g_1(T_1) = T_1$ and $g_r \circ \cdots g_1$ restricted to F_1 is the identity.

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Theorem

Let D be a compact Poincaré polyhedron in $\mathrm{H}^n_{\mathbb{C}}$ with side-pairing transformations Δ satisfying condition **Cyclic**. *Let Γ be the group generated by Δ. Then Γ is a discrete subgroup of $\mathrm{Isom}(\mathrm{H}^n_{\mathbb{C}})$, D is a fundamental domain for Γ and Γ has presentation:*

$$\Gamma = \langle \Delta \mid \textit{cycle relations, reflection relations} \rangle$$